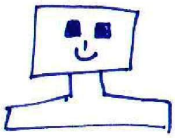
4 colour Theorem

Something that is definitely true.

You only need 4 colours to colour any map, following the rule that touching regions are different colours.

Proof = Explanation showing something is definitely true.



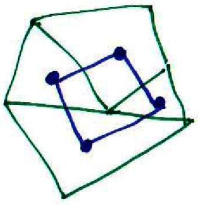
Computers were needed to show this.

Little man got quite tired.

This is the little man's biscuit tin, because calculating is very tiring.

SEVEN COLOUR THEOREM

Sometimes, switching maps to graphs is a good idea.



graph

Map

EXCITING NEW GAME:

Colour vertices so connected vertices are different.

Actually it is the same game... sorry.

Euler's Formula:

Any graph: $V - E + F = 1$

V = # of vertices
E = # of edges
F = # of faces.

Even more exciting actually new game.....

→ Drawing graphs on CURVED surfaces!

On a sphere, graphs get an extra face. This is like the outside of the graph

So, now ~~a~~ magic formula is:

$$V - E + F = 2$$

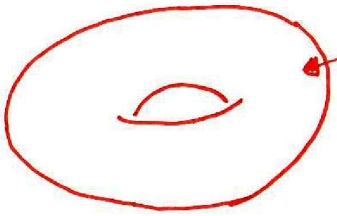
[use the same key as before].

You can travel between these two exciting worlds. The graph colouring problem, therefore, stays the same (when on a ball).

Maybe this wasn't such an exciting game ☹

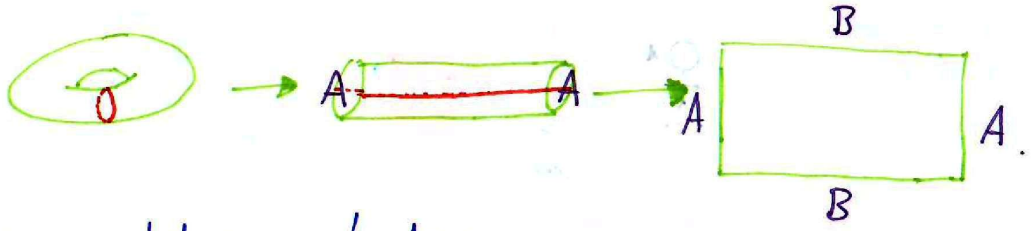
But other surfaces exist, e.g. a doughnut (aka a TORUS)

This game is actually different, because you can go through holes.



I'm not very good at drawing torusses.

so you can do this:

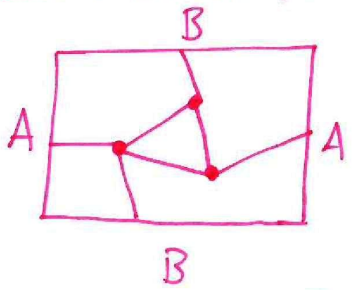


These are a bit easier to draw.

If you walk through one side, you get to the otherside.

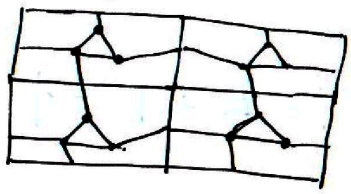
What happens to Euler's formula?

Look at an example:



$N=3$
$E=5$
$F=2$

To count faces, tesselate.



so $V-E+F=0$?

But what about things like this?

Here $V-E+F=2$!!!

That is so sad that we might have to ignore these graphs. Since one of the faces has a whole, so this isn't allowed.

Seven colour theorem:

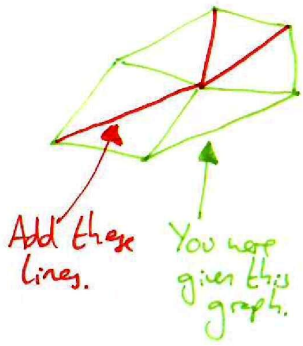
→ You only need 7 colours to colour a graph on the torus.

Of course, you ignore the sad graph from above.

Now for exciting proofs. ☺

STEP 1:

If you are given a graph, then adding new edges to make things into triangles.



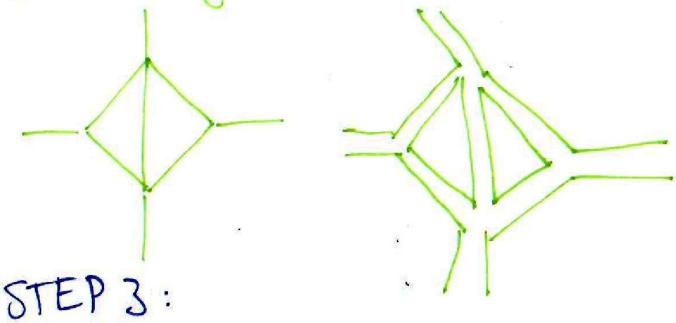
This is called a TRIANGULATION.

→ Counting is easier in a triangulation.

→ So that is why they are good.

STEP 2:

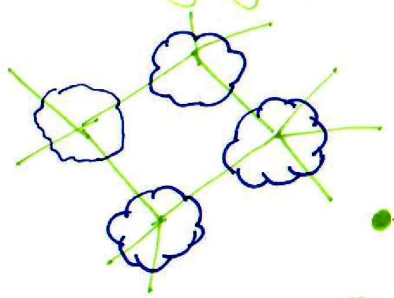
On a triangulation, $3F = 2E$.



STEP 3:

Count faces in a different way:

Take any graph:



Definition: Number of edges hitting an vertex is the degree.

- $2E = \sum_{\text{vertex}} \text{degree of vertices}$

[each edge adds one to the degree of 2 vertices]

STEP 4:

Suppose there is a triangulation on a torus.

What do we know?

1) $2E = 3E$
2) $2E = \sum \text{degrees}$
3) $V - E + F = 0$

$\Longrightarrow \frac{1}{V} \sum \text{degrees} = 6$ [magic algebra].

This should show that

$3F = \sum \text{degrees}$

$\Rightarrow 6F = 2\sum \text{degrees}$ ← Magic, slightly uninteresting algebra stuff.

$\Rightarrow 6E = 3\sum \text{degrees}$

$\Rightarrow 6V = 6E - 6F = \sum \text{degrees}.$

STEP 5: [This is VERY IMPORTANT]

Since the average number of degrees [horrible wording, sorry!] on a triangulation is 6, we can do some maths.

Take a graph and make it into a triangulation.

* If all vertices had degree above 6, the average couldn't be 6.
* So some vertex must have degree at most 6.
* And you only added edges at the beginning...

$\Rightarrow$ In any graph on a torus, some vertex has degree at most 6.

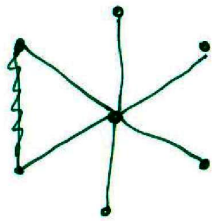
STEP 6:

If the graph has at most 7 vertices, then everything is good ☺ and very easy.

Suppose the graph has 8 vertices.

It is on a torus, so:

One vertex looks a bit like this



Now ignore that vertex.

Sorry little vertex.

Now colour in the rest of the graph.

And then colour in that vertex, which you can do, because it only has 6 neighbours.

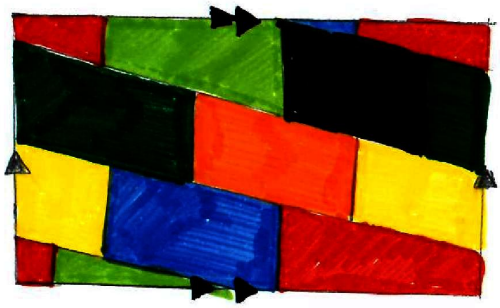
If the graph has 9 vertices, then do the same thing, to get an 8 vertex graph, which you can do. Then colour in your final vertex.

A similar argument holds for 10, and 11, and 12, and so on.

Therefore, ALL graphs can be coloured in at most 7 colours! Yay!

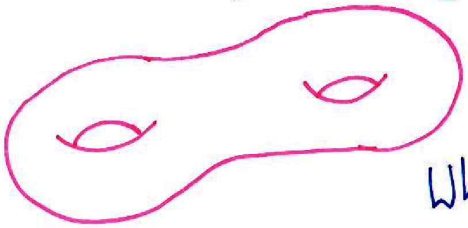
Are 7 colors necessary? Or can you always do it in 6?

7 colours ARE necessary, because there is an example that needs it:



What next?

Why not other surfaces? E.g. Double Doughnuts.



How many colours do you need now?

What is Euler's formula here?